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THE SINGULARITIES OF YANG-MILLS CONNECTIONS FOR BUNDLES ON A SURFACE. II. THE STRATIFICATION

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Abstract. Let Σ be a closed surface, G a compact Lie group, not necessarily connected, with Lie algebra g, endowed with an adjoint action invariant scalar product, let $\xi\colon P \to \Sigma$ be a principal G-bundle, and pick a Riemannian metric and orientation on Σ so that the corresponding Yang-Mills equations are defined. In an earlier paper we determined the local structure of the moduli space $N(\xi)$ of central Yang-Mills connections on ξ near an arbitrary point. Here we show that the decomposition of $N(\xi)$ into connected components of orbit types of central Yang-Mills connections is a stratification in the strong (i. e. Whitney) sense; furthermore each stratum, being a smooth manifold, inherits a finite volume symplectic structure from the given data. This complements, in a way, results of Atiyah-Bott in that it will in general decompose further the critical sets of the corresponding Yang-Mills functional into smooth pieces.

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Introduction

Let Σ be a closed surface, G a compact Lie group, with Lie algebra g, and ξ a principal G-bundle over Σ, having a connected total space P. Further, pick a Riemannian metric on Σ and an *orthogonal structure* on g, that is, an adjoint action invariant positive definite inner product. These data give rise to a Yang-Mills theory studied in great detail in [4] for a connected structure group. We assume that solutions of the corresponding Yang-Mills equations exist — this will always be the case for a connected structure group, cf. [4] — and we denote the moduli space of central Yang-Mills connections by $N(\xi)$. In an earlier paper [24] to which we refer for background and notation we determined the local structure of $N(\xi)$ near the class of an arbitrary central Yang-Mills connection. Extending the approach in [24] we establish here the following.

Theorem. *The decomposition of $N(\xi)$ into connected components of orbit types of central Yang-Mills connections is a Whitney stratification; furthermore each stratum, being a smooth manifold, inherits a symplectic structure from the given data with finite symplectic volume.*

A more precise statement will be given in (1.2) below, and the finiteness of the symplectic volumes will be proved in (1.7) below. The corresponding decomposition of $N(\xi)$ into pieces will be given in (1.1). In a way, the Theorem complements and extends results of Atiyah-Bott [4]; in fact, our stratification will in general decompose further each component of the critical set of the Yang-Mills functional. In particular, for $G = U(n)$, the unitary group, in the "coprime case", cf. Atiyah-Bott [4], the moduli space of central Yang-Mills connections is smooth, and our stratification then consists of a single piece.

In a follow up paper [8] we identify the strata mentioned in the Theorem with reductions to suitable subbundles. Thereby we *cannot* avoid running into principal bundles with *non-connected* structure groups, even when the structure group of the bundle ξ we started with is connected. This is the reason why the present theory has been set up for general compact not necessarily connected structure groups.

It is known that the decomposition of a moduli space of connections according to orbit types is a manifold decomposition, cf. Kondracki-Sadowski [25], referred to sometimes as a stratification. The result given in the above Theorem is really different and considerably stronger: The space $\mathcal{N}(\xi)$ of central Yang-Mills connections is *not* a smooth submanifold of the space of all connections on ξ; moreover our Theorem says in particular that the decomposition into connected components of orbit types is a stratification in the strong sense, that is, a certain additional "cone condition" also holds.

The reader is assumed familiar with our paper [24]. Notations and definitions given there will not be repeated.

1. The stratification

We shall use the notion of *stratified space* in the sense of Goresky-Mac Pherson [23]. For a closed subgroup $K \subseteq G$ let $\mathcal{A}_{(K)} \subseteq \mathcal{A}(\xi)$ denote the subspace of all connections A having stabilizer Z_A whose image in G is conjugate to K. We take as indexing set $\mathcal{I}$ the set of all possible stabilizer subgroups of central Yang-Mills connections A

on ξ modulo conjugacy. For each $(K) \in \mathcal{I}$, let

$$\mathcal{N}_{(K)} = \mathcal{N}(\xi) \cap \mathcal{A}_{(K)}, \quad N_{(K)} = \mathcal{N}_{(K)}/\mathcal{G}(\xi).$$

As far as conjugacy classes are concerned it suffices to take as representatives of the elements of $\mathcal{I}$ conjugacy classes of subgroups of G rather than $\mathcal{G}(\xi)$, and we shall henceforth do so. In particular we shall write $(K) < G$, $(K) \in \mathcal{I}$ etc. The space $\mathcal{N}(\xi)$ of central Yang-Mills connections decomposes into a disjoint union of *orbit types* $\mathcal{N}_{(K)}$, $(K) \in \mathcal{I}$, and accordingly the moduli space $N(\xi)$ of central Yang-Mills connections decomposes into a disjoint union

$$N(\xi) = \bigcup_{(K) \in \mathcal{I}} N_{(K)} \tag{1.1}$$

of *orbit types*. The decomposition we are after is (1.1), with each *piece* $N_{(K)}$ decomposed further into its connected components. We shall not distinguish in notation between (1.1) and its refinement into connected components.

Theorem 1.2. *The decomposition of $N(\xi)$ into connected components of the pieces of* (1.1) *is a stratification; furthermore each piece $N_{(K)}$, being a finite union of smooth manifolds, inherits a symplectic structure $\sigma_{(K)}$ whose pull back to $\mathcal{N}_{(K)} \subseteq \mathcal{A}(\xi)$ equals the restriction to $\mathcal{N}_{(K)}$ of the symplectic form σ on $\mathcal{A}(\xi)$.*

Addendum. *The decomposition of $N(\xi)$ into connected components of the pieces of* (1.1) *is a Whitney stratification.*

Proof. Clearly the issue is local. We shall establish the theorem by combining a result given in SJAMAAR [20] (3.5) and in Section 6 of SJAMAAR-LERMAN [21] with the local analysis of $N(\xi)$ near a point $[A]$ given in Section 2 of our paper [24]. We proceed as follows.

Let A be a central Yang-Mills connection on ξ fixed henceforth. Suppose the smooth finite dimensional symplectic submanifold $\mathcal{M}_A$ of $\mathcal{A}(\xi)$ chosen as in the proof of [24] (2.32). We maintain the notation $\mathcal{N}_A = \mathcal{N}(\xi) \cap \mathcal{M}_A$ and $N_A = \mathcal{N}_A/Z_A$ introduced in Section 2 of [24]. The injection of $\mathcal{M}_A$ into $\mathcal{A}(\xi)$ induces an injection of N_A into $N(\xi)$ which is in fact a homeomorphism of N_A onto a neighborhood U (say) of $[A]$ in $N(\xi)$. This neighborhood U, in turn, inherits a structure of a decomposed space from the decomposition (1.1) of $N(\xi)$. On the other hand, with respect to the Z_A-structure on $\mathcal{N}_A$ inherited from that on $\mathcal{M}_A$, the space N_A decomposes into connected components of orbit types, and the injection of N_A into $N(\xi)$ is decomposition preserving.

Let $\vartheta_A \colon \mathcal{M}_A \to z_A^*$ be the momentum mapping [24] (2.30) for the hamiltonian Z_A-action on $\mathcal{M}_A$; it has the value zero at the point A. It is then manifest that the space N_A coincides with the *Marsden-Weinstein* reduced space $\vartheta_A^{-1}(0)/Z_A$. Consequently, in view of the main result of SJAMAAR-LERMAN [21], applied to the momentum mapping ϑ_A for the *compact* group Z_A, the decomposition of N_A into connected components of orbit types is a stratification in such a way that each piece or stratum inherits a symplectic structure from the given data. This establishes the theorem since the injection of N_A into $N(\xi)$ is decomposition preserving. □

Remark 1.3. Let Θ_A be the momentum mapping from $\mathrm{H}^1_A(\Sigma, \mathrm{ad}(\xi))$ to $\mathrm{H}^2_A(\Sigma, \mathrm{ad}(\xi))$ given in [24] (1.2.5), for the Z_A-action on $\mathrm{H}^1_A(\Sigma, \mathrm{ad}(\xi))$; it is given by the assignment to $\eta \in \mathrm{H}^1_A(\Sigma, \mathrm{ad}(\xi))$ of $\frac{1}{2}[\eta, \eta]_A \in \mathrm{H}^2_A(\Sigma, \mathrm{ad}(\xi))$, where $[\cdot, \cdot]_A$ refers to the graded bracket on $\mathrm{H}^*_A(\Sigma, \mathrm{ad}(\xi))$ induced by the data. By (2.32) of [24], the reduced space $\mathrm{H}_A = \Theta_A^{-1}(0)\big/Z_A$ is a local model of a neighborhood of the point $[A]$ of $N(\xi)$. The theorem can also be established by means of a direct argument applied to this local model, of the kind used in Sjamaar [20] and Sjamaar-Lerman [21] in the finite dimensional setting.

Proof of the Addendum. In Section 3 of Sjamaar [20] and (6.5) of Sjamaar-Lerman [21] it is proved that a space of the kind H_A may be embedded into a euclidean space as a *Whitney* stratified set. This implies that (1.1) is a Whitney stratification. □

The stratification (1.1) of $N(\xi)$ has a number of remarkable properties:

Theorem 1.4. (1) *The link of each point of $N(\xi)$, if non-empty, is connected.*
(2) *There is a unique connected stratum $N_{(H)}$ which is open and dense in $N(\xi)$.*

We shall refer to the unique open, connected and dense stratum as the *top* stratum of $N(\xi)$, written $N^{\mathrm{top}}(\xi)$.

Proof. Theorem 1.4 is established in virtually the same way as the corresponding result for the reduced space arising from a momentum mapping for the action of a compact group on a smooth finite dimensional manifold due to Sjamaar [20] (3.4.9) and Sjamaar-Lerman [21] (5.9). The only difference is that under the present circumstances we know *a priori* that the reduced space is *connected*. We leave the details to the reader. □

The following consequence of (1.4(2)) seems to us to deserve special mention.

Corollary 1.5. *There is a closed subgroup T of G, unique up to conjugacy, such that the space of gauge equivalence classes of central Yang-Mills connections A having orbit type (T) coincides with $N^{\mathrm{top}}(\xi)$.* □

Maintaining terminology introduced in [24], we shall say that a central Yang-Mills connection A is *non-singular* if its stabilizer Z_A acts trivially on $\mathrm{H}^1_A(\Sigma, \mathrm{ad}(\xi))$; the point $[A]$ of $N(\xi)$ will then be said to be *non-singular*. In view of [24] (2.33), a non-singular central Yang-Mills connection A represents a *smooth* point of $N(\xi)$ in the sense that its link is a sphere. It may happen that the subspace of smooth points of $N(\xi)$ is larger than that of its non-singular ones; for example this occurs for $G = \mathrm{SU}(2)$ over a surface of genus 2, see [12]. However the subspace of non-singular points is exactly that where the symplectic structure is defined.

Corollary 1.6. *The space of gauge equivalence classes of non-singular central Yang-Mills connections A is non-empty and coincides with the top stratum $N^{\mathrm{top}}(\xi)$.*

Proof. Let A be a central Yang-Mills connection. By (2.32) of [24], a neighborhood of $[A]$ in $N(\xi)$ looks like a neighborhood of the class of zero in the reduced space H_A. When $[A]$ lies in the top stratum of $N(\xi)$ the class of 0 lies in the top stratum of H_A, viewed as a stratified symplectic space. However this is possible only if Z_A acts trivially on $\mathrm{H}^1_A(\Sigma, \mathrm{ad}(\xi))$. □

Theorem 1.7. *Each stratum of $N(\xi)$ has finite symplectic volume.*

Proof. Let A be a central Yang-Mills connection, let $\widetilde{B}_A \subseteq \mathrm{H}^1_A(\Sigma, \mathrm{ad}(\xi))$ be an open relatively compact Z_A-invariant ball centered at the origin, and let B_A be the reduced space for the restriction of the momentum mapping Θ_A to $\widetilde{B}_A$. As stratified space with symplectic strata, B_A is a local model for $N(\xi)$ near the point represented by A, cf. (1.3) above. The argument for (3.9) in [21] is valid for a space of the kind B_A and shows that each stratum of B_A has finite symplectic volume. This implies the statement since $N(\xi)$ may be covered by finitely many open sets having a model of the kind B_A. $\square$

2. The relationship with representations

We pick a base point Q of Σ. Let $0 \to \mathbf{Z} \to \Gamma \to \pi \to 1$ be the universal central extension of the fundamental group $\pi = \pi_1(\Sigma, Q)$ of Σ. It arises from the standard presentation

$$\mathcal{P} = \langle x_1, y_1, \ldots, x_\ell, y_\ell; r \rangle, \quad r = \prod_{j=1}^{\ell} [x_j, y_j], \tag{2.1}$$

of π as follows, the number ℓ being the genus of Σ: Let F be the free group on the generators and N the normal closure of r in F; then $\Gamma = F\big/[F,N]$. Let $\Gamma_{\mathbf{R}}$ be the group obtained from Γ when its centre $\mathbf{Z}$ is extended to the additive group $\mathbf{R}$ of the reals. It plays a certain universal role which we now explain:

Write S^1 for the circle group, and let $\mu\colon M \to \Sigma$ be the unique principal S^1-bundle having Chern class 1. We endow it with a harmonic or Yang-Mills connection A_Σ having (normalized) constant curvature $2\pi i \mathrm{vol}_\Sigma$. The universal covering projection $\widetilde{M} \to M$, combined with μ, yields the projection map $\mu^\sharp\colon \widetilde{M} \to \Sigma$ of a principal $\Gamma_{\mathbf{R}}$-bundle together with a morphism of principal bundles from $\mu^\sharp$ to μ; in particular, it induces a surjective homomorphism from $\Gamma_{\mathbf{R}}$ to S^1 inducing an isomorphism of Lie algebras. Let $A^\sharp_\Sigma$ be the lift of A_Σ to $\mu^\sharp$. It is a central Yang-Mills connection on $\mu^\sharp$, with reference to the obvious bi-invariant metric on $\Gamma_{\mathbf{R}}$. Let $\Omega\Sigma$ denote the space of piecewise smooth loops in Σ, having starting point Q; with the usual composition of loops, parametrized by intervals of arbitrary length, $\Omega\Sigma$ is an *associative* topological monoid.

We now return to our principal bundle ξ. We pick base points $\widehat{Q} \in P$ and $\widetilde{Q} \in \widetilde{M}$ over the base point Q of Σ. For every connection A on ξ, parallel transport furnishes a continuous homomorphism $\tau_{A,\widehat{Q}}$ of monoids from $\Omega\Sigma$ to G. In particular, $A^\sharp_\Sigma$ induces the homomorphism $\tau_{A^\sharp_\Sigma,\widetilde{Q}}$ from $\Omega\Sigma$ to $\Gamma_{\mathbf{R}}$. Moreover, cf. [9] (2.3), a Yang-Mills connection A on ξ induces a homomorphism $\chi_{A,\widehat{Q}}$ from $\Gamma_{\mathbf{R}}$ to G so that

$$\tau_{A,\widehat{Q}} = \chi_{A,\widehat{Q}} \circ \tau_{A^\sharp_\Sigma,\widetilde{Q}}\colon \Omega\Sigma \to G,$$

whence the assignment to a central Yang-Mills connection A of the restriction of the homomorphism $\chi_{A,\widehat{Q}}$ to Γ yields a map from $\mathcal{N}(\xi)$ to $\mathrm{Hom}(\Gamma, G)$ the image of which we denote by $\mathrm{Hom}_\xi(\Gamma, G)$. The latter is compact and closed under conjugation, and

we write $\mathrm{Rep}_\xi(\Gamma, G) = \mathrm{Hom}_\xi(\Gamma, G)/G$. When ξ is flat the space $\mathrm{Hom}_\xi(\Gamma, G)$ amounts to the subspace $\mathrm{Hom}_\xi(\pi, G)$ of $\mathrm{Hom}(\pi, G)$ corresponding to ξ, and the same kind of remark can be made for $\mathrm{Rep}_\xi(\Gamma, G)$. We maintain the notation $\mathcal{G}^Q(\xi)$ for the group of *based* gauge transformations. With these preparations out of the way we recall the following, cf. Section 2 in our paper [9] for more details.

Theorem 2.3. *The assignment to a central Yang-Mills connection A of $\chi_{A,\widehat{Q}}$ induces a homeomorphism from $\mathcal{N}(\xi)/\mathcal{G}^Q(\xi)$ onto* $\mathrm{Hom}_\xi(\pi, G)$ *compatible with the G-actions and hence a homeomorphism*

$$N(\xi) \to \mathrm{Rep}_\xi(\Gamma, G). \tag{2.3.1}$$

In view of the identification (2.3.1) of $\mathrm{Rep}_\xi(\Gamma, G)$ with $N(\xi)$, the stratification (1.1) of $N(\xi)$ passes to a decomposition of $\mathrm{Rep}_\xi(\Gamma, G)$. The corresponding pieces of the resulting decomposition of $\mathrm{Rep}_\xi(\Gamma, G)$ are just as well orbit types, in view of the following the proof of which we leave to the reader.

Lemma 2.4. *For every connection A on ξ, the projection map from $\mathcal{G}(\xi)$ to G (determined by the choice of base point $\widehat{Q}$) identifies the stabilizer Z_A of A with the stabilizer of its class $[A]$ in $\mathcal{A}(\xi)/\mathcal{G}^Q(\xi)$.* $\square$

The Lemma entails that the bijection (2.3.1) preserves the decompositions into orbit types on both sides.

Under favorable circumstances, for example for connected structure group, the top stratum of the stratification (1.1) of $N(\xi)$ can be described by representation theory. Following RAMANATHAN [27], we shall say that a representation $\chi\colon L \to G$ of a group L is *irreducible* if the subspace g^L of L-invariants in g under the composite of χ with the adjoint representation of G on g coincides with the Lie algebra z of the centre Z of G. Now, for an arbitrary central Yang-Mills connection A, evaluation of a section of $\mathrm{ad}(\xi)$ at Q induces an isomorphism from $\mathrm{H}^0_A(\Sigma, \mathrm{ad}(\xi))$ onto the subspace $g^\Gamma \subseteq g$ of invariants in g, with reference to the induced homomorphism $\chi_{A,\widehat{Q}} \in \mathrm{Hom}_\xi(\Gamma, G)$, and hence $\chi_{A,\widehat{Q}}$ is manifestly irreducible if and only if this association identifies $\mathrm{H}^0_A(\Sigma, \mathrm{ad}(\xi))$ with z. We shall say that a central Yang-Mills connection A is *representation irreducible* if its stabilizer Lie algebra $z_A = \mathrm{H}^0_A(\Sigma, \mathrm{ad}(\xi))$ is identified with z in this way. We have chosen this terminology in order to avoid conflict with the common notion of an *irreducible* connection. When G is connected and the genus of Σ is at least 2, (7.1) and (7.7) of RAMANATHAN [27] imply that there exist irreducible representations in $\mathrm{Hom}_\xi(\Gamma, G)$ and hence representation irreducible central Yang-Mills connections on ξ.

For a representation irreducible central Yang-Mills connection A on ξ, the stabilizer Lie algebra z_A amounts to the Lie algebra z of the centre Z of G and the latter lies in the centre of the Lie algebra g; consequently the action of the stabilizer Z_A of A on $\mathrm{H}^1_A(\Sigma, \mathrm{ad}(\xi))$ then passes to a representation of the finite group $\pi_0(Z_A)$ of connected components of Z_A on $\mathrm{H}^1_A(\Sigma, \mathrm{ad}(\xi))$.

Theorem 2.5. *Suppose that representation irreducible central Yang-Mills connections exist. Then the top stratum N^{top} consists exactly of the classes $[A]$ of representation irreducible central Yang-Mills connections having the property that the induced*

representation of the finite group of connected components $\pi_0(Z_A)$ *on* $\mathrm{H}^1_A(\Sigma, \mathrm{ad}(\xi))$ *is trivial, whatever representative* A *of* $[A]$.

Proof. Let A be a representation irreducible central Yang-Mills connection. Then the stabilizer Z_A contains an isomorphic copy of the centre Z of G and has Lie algebra z_A isomorphic to the Lie algebra z of Z. Moreover, cf. Section 2 in [24], the momentum mapping Θ_A from $\mathrm{H}^1_A(\Sigma, \mathrm{ad}(\xi))$ to $\mathrm{H}^2_A(\Sigma, \mathrm{ad}(\xi))$ is zero. Hence a neighborhood of $[A]$ in $N(\xi)$ looks like a neighborhood of the image of zero in the quotient space $\mathrm{H}^1_A(\Sigma, \mathrm{ad}(\xi))\big/\pi_0(Z_A)$ modulo the induced action of the finite group $\pi_0(Z_A)$ on $\mathrm{H}^1_A(\Sigma, \mathrm{ad}(\xi))$. Consequently, cf. (1.6), the point $[A]$ of $N(\xi)$ is non-singular if and only if the representation of the finite group $\pi_0(Z_A)$ on $\mathrm{H}^1_A(\Sigma, \mathrm{ad}(\xi))$ is trivial. □

As observed by ATIYAH-BOTT [4] for the case of a connected structure group, Theorem 2.3 may be used to reduce the study of central Yang-Mills connections to that of flat connections. In fact, write Z_e for the connected component of the identity of the centre of G, and let $G^\sharp = G/Z_e$. Let

$$\xi^\sharp \colon P^\sharp = P/Z_e \longrightarrow \Sigma \tag{2.6}$$

be the induced principal $G^\sharp$-bundle arising from dividing out the group Z_e. We then have the following.

Theorem 2.7. *The bundle* $\xi^\sharp$ *is flat, the central Yang-Mills connections on* $\xi^\sharp$ *are precisely the flat ones, and the map from* $N(\xi)$ *to* $N(\xi^\sharp)$ *induced by the obvious morphism of principal bundles from* ξ *to* $\xi^\sharp$ *is in fact the projection map of a principal fibre bundle having compact connected structure group* $Z_e^{2\ell} = \mathrm{Hom}(\pi, Z_e)$. *Moreover this map is compatible with the stratifications.*

In particular, it suffices to study the stratification of $N(\xi^\sharp)$.

Proof. The first statement has been established in (3.10) of our paper [9]. It is clear that the bundle map is compatible with the stratifications, in view of the naturality of the constructions. □

EXAMPLE 2.8. In view of an observation of RAMANATHAN [27], it may well happen that the class of a representation irreducible central Yang-Mills connection A is still a singular point in the moduli space. His example is the following one, cf. (4.1) in [27]: Let H be the subgroup of $\mathrm{SO}(n)$ consisting of diagonal matrices with entries ± 1; it is irreducible in the sense that the space of H-invariants in the Lie algebra $so(n)$ under the adjoint action is zero. When the genus p of Σ is sufficiently large, e. g. $2\ell > 2^n$, there is a surjective homomorphism χ from π to H. The associated flat connection $A = A_\chi$ on the resulting flat principal $\mathrm{SO}(n)$-bundle is representation irreducible by definition. The stabilizer Z_A of $A = A_\chi$ coincides with H. Now $\mathrm{H}^1_A(\Sigma, \mathrm{ad}(\xi))$ is canonically isomorphic to $\mathrm{H}^1(\pi, g_\chi)$ and the resulting representation of $H = \pi_0(H)$ on $\mathrm{H}^1(\pi, g_\chi)$ is non-trivial.

When Σ is a 2-torus there are no irreducible representations of π in G and hence no representation irreducible central Yang-Mills connections on ξ unless G is abelian; for G connected non-abelian, the top stratum of $N(\xi)$ then corresponds to the conjugacy class (T) of a maximal torus T in G.

3. Examples

Over the 2-sphere the theory is not interesting since $N(\xi)$ then consists of a single point. Until further notice we assume G connected. We denote by H the connected component of the identity of the centre of G, and we write ℓ for the genus of Σ.

3.1. Genus 1

The fundamental group $\pi = \pi_1(\Sigma)$ is abelian and there are no representation irreducible central Yang-Mills connections on ξ unless G is abelian, which we exclude henceforth. Let T be a maximal torus in G/H, and let W be its Weyl group. It is well known that the obvious injection of $\mathrm{Hom}(\pi, T)$ into $\mathrm{Hom}_{\xi^\sharp}(\pi, G/H)$ identifies the space $(T \times T)/W$, the W-action on $T \times T$ being the diagonal one, with the representation space $\mathrm{Rep}_{\xi^\sharp}(\pi, G/H)$; see (2.6) for the notation $\xi^\sharp$. Consequently $N(\xi)$ is the total space of a principal $H \times H$-fibre bundle over $(T \times T)/W$ as base. Now $(T \times T)/W$, being a V-manifold, is stratified in the usual way.

3.2. Genus ≥ 2; $G = \mathrm{SU}(2)$

Since the group $G = \mathrm{SU}(2)$ is simply connected there is only the trivial $\mathrm{SU}(2)$-bundle ξ over Σ. Consider the space $\mathrm{Hom}(\pi, \mathrm{SU}(2))$ as a subspace of $G^{2\ell}$ in the usual way, and let T be the standard circle subgroup inside G; it is a maximal torus. Consider the space $\mathrm{Hom}(\pi, T)$, viewed as a subspace of $\mathrm{Hom}(\pi, G)$; clearly, the former looks like $T^{2\ell}$. Let Y be the G-orbit of $\mathrm{Hom}(\pi, T)$ in $\mathrm{Hom}(\pi, G)$ under the adjoint action. Then $\mathrm{Hom}(\pi, \mathrm{SU}(2))$ decomposes into $\mathrm{Hom}(\pi, \mathrm{SU}(2)) \setminus Y$ and Y. Each point in $\mathrm{Hom}(\pi, \mathrm{SU}(2)) \setminus Y$ has stabilizer the centre $Z = \{\pm 1\}$, that is, is an irreducible representation while each point in $\mathrm{Hom}(\pi, T) \cong T^{2\ell}$ has stabilizer T. Furthermore the inclusion of $\mathrm{Hom}(\pi, T)$ into Y induces a bijection of $\mathrm{Hom}(\pi, T)/W$ onto Y/G where $W = \mathbf{Z}/2$ refers to the Weyl group of $\mathrm{SU}(2)$. Now the non-trivial element w of W acts on $\mathrm{Hom}(\pi, T)$, viewed as $T^{2\ell}$, by the assignment to $(\zeta_1, \ldots, \zeta_{2\ell}) \in T^{2\ell}$ of $(\overline{\zeta}_1, \ldots, \overline{\zeta}_{2\ell})$ where as usual $\overline{\zeta}$ refers to the complex conjugate of $\zeta \in T$. Hence the fixed point set N_G of the action of W on $\mathrm{Hom}(\pi, T)$ consists of the homomorphisms ϕ having the values ± 1 on the generators of π spelled out in (2.1), and the W-action is free on $\mathrm{Hom}(\pi, T) \setminus N_G$. Thus the resulting stratification looks like

$$N(\xi) = N_G \cup N_{(T)} \cup N_Z.$$

Here $N_Z = (\mathrm{Hom}(\pi, G) \setminus Y)/G$ and $N_{(T)} = (Y \setminus N_G)/G$. Moreover the projection map from $\mathrm{Hom}(\pi, G) \setminus Y$ to N_Z is actually a principal $\mathrm{SO}(3)$-bundle map whence N_Z is a smooth manifold of dimension $6\ell - 6$. Furthermore $N_{(T)}$ is manifestly a connected smooth manifold of dimension 2ℓ. This example is studied in more detail in our papers [10] and [12].

3.3. The trivial $\mathrm{SO}(3)$-bundle over a surface of genus ≥ 2

Since the group $G = \mathrm{SO}(3)$ has $\pi_1(G) = \mathbf{Z}/2$ there is the trivial G-bundle and a single non-trivial one. Consider first the trivial one say ξ. The obvious projection map from $\mathrm{SU}(2)$ to $\mathrm{SO}(3)$ induces a covering projection from $\mathrm{Hom}(\pi, \mathrm{SU}(2))$ onto $\mathrm{Hom}_\xi(\pi, \mathrm{SO}(3))$ having $\mathrm{Hom}(\pi, \mathbf{Z}/2)$ as its group of deck transformations, that is, the group $(\mathbf{Z}/2)^{2\ell}$. The latter passes to a covering projection from $\mathrm{Rep}(\pi, \mathrm{SU}(2))$ onto $\mathrm{Rep}_\xi(\pi, \mathrm{SO}(3))$ having still $(\mathbf{Z}/2)^{2\ell}$ as its group of deck transformations. Under this covering, in turn, the stratification

$$N(\widetilde{\xi}) = N_{\widetilde{G}} \cup N_{(\widetilde{T})} \cup N_Z$$

of $N(\widetilde{\xi})$ for the trivial SU(2)-bundle $\widetilde{\xi}$ obtained in (3.2) above (and written ξ there) passes to the stratification

$$N(\xi) = N_G \cup N_{(T)} \cup N_e;$$

here $\widetilde{T}$ and T refer to the corresponding circle groups inside $\widetilde{G} = \mathrm{SU}(2)$ and $G = \mathrm{SO}(3)$, respectively.

3.4. Genus ≥ 2; $G = \mathrm{U}(2)$

The connected component H of the identity of the centre of $U(2)$ coincides with the centre S^1 and the quotient G/H is the group SO(3). A principal U(2)-bundle ξ is classified by its Chern class. Moreover the corresponding SO(3)-bundle $\xi^\sharp$ is trivial or not according as the Chern class is even or not. Hence, for even Chern class, in view of (2.7) above, $N(\xi)$ is the total space of a principal $(S^1)^{2\ell}$-bundle over $N(\xi^\sharp)$, and the stratification of $N(\xi^\sharp)$ obtained in (3.3) above induces the stratification of $N(\xi)$. It is well known, cf. Narasimhan-Seshadri [18], that, for odd Chern class, the space $N(\xi)$ is a smooth compact manifold.

3.5. The non-trivial SO(3)-bundle over a surface of genus ≥ 2

Let ξ be the non-trivial principal SO(3)-bundle over Σ, and write $\widetilde{\xi}$ for a principal U(2)-bundle having odd Chern class. In view of (2.7) above, $N(\widetilde{\xi})$ is the total space of a principal $(S^1)^{2\ell}$-bundle over $N(\xi)$. Since $N(\widetilde{\xi})$ is a smooth compact manifold so is $N(\xi)$.

3.6. Genus ≥ 1; $G = \mathrm{O}(2)$

The group $G = \mathrm{O}(2)$ is a semi-direct product $G = S^1 \times_s \mathbf{Z}/2$. Principal O(2)-bundles ξ over Σ having a connected total space P are classified as follows: At first, inspection of its homotopy exact sequence reveals that ξ determines a unique homomorphism ϕ_ξ from π to $\mathbf{Z}/2$ which, since P is assumed connected, has to be *non-trivial*. Now, given a non-trivial homomorphism ϕ from π to $\mathbf{Z}/2$, there is always a principal bundle ξ so that $\phi = \phi_\xi$, and for fixed ϕ, the principal O(2)-bundles having $\phi_\xi = \phi$ are classified by the cohomology group $\mathrm{H}^2_\phi(\Sigma, \pi_1(\mathrm{SO}(2))$, where the subscript ϕ refers to local coefficients with respect to the induced (non-trivial) $\mathbf{Z}/2$-action on $\pi_1(\mathrm{SO}(2)) \cong \mathbf{Z}$. A non-trivial homomorphism ϕ from π to $\mathbf{Z}/2$ may be realized by a map from Σ to $\mathbf{R}P_2$ inducing an isomorphism of $\mathrm{H}^2_{\mathrm{local}}(\mathbf{R}P_2, \mathbf{Z})$ onto $\mathrm{H}^2_\phi(\Sigma, \pi_1(\mathrm{SO}(2))$. Under this map we obtain *every* principal O(2)-bundle on Σ having $\phi_\xi = \phi$ induced from a principal O(2)-bundle on $\mathbf{R}P_2$. In particular, we see that the cohomology group $\mathrm{H}^2_\phi(\Sigma, \pi_1(\mathrm{SO}(2))$ classifies these bundles.

We now pick a principal O(2)-bundle ξ on $\mathbf{R}P_2$ having $\phi_\xi = \phi$ and consider the subspace $\mathrm{Hom}_\xi(\pi, \mathrm{O}(2))$ of $\mathrm{Hom}(\pi, \mathrm{O}(2))$ which corresponds to ξ. It admits the following description. We denote the two elements of $\mathbf{Z}/2$ by 1 and -1, the latter being the non-trivial element. Write $G_1 = \mathrm{SO}(2)$, write G_{-1} for the other connected component of O(2) and, with reference to the presentation (2.1), consider the subspace

$$G_{\phi(x_1)} \times G_{\phi(y_1)} \times \cdots \times G_{\phi(x_\ell)} \times G_{\phi(y_\ell)}$$

of $G^{2\ell}$. Then the space $\mathrm{Hom}_\xi(\pi, \mathrm{O}(2))$ appears as that of 2ℓ-tuples $(u_1, v_1, \ldots, u_\ell, v_\ell)$ in this subspace having the property $\prod [u_j, v_j] = 1$. Now, given a point ψ of

$\mathrm{Hom}_\xi(\pi, \mathrm{O}(2))$, the Lie algebra of its stabilizer amounts to the cohomology group $\mathrm{H}^0(\pi, g_\psi)$. However, under the present circumstances, $g = \mathbf{R}$, and the action of π on $g = \mathbf{R}$ factors through the given non-trivial homomorphism ϕ from π to $\mathbf{Z}/2$ and hence is non-trivial. Consequently $\mathrm{H}^0(\pi, g_\psi) = \mathrm{H}^0(\mathbf{Z}/2, g_\psi) = 0$, and the stabilizer of any point of $\mathrm{Hom}_\xi(\pi, \mathrm{O}(2))$, if non-trivial, is a finite group. This implies that $N(\xi)$ is a symplectic V-manifold of dimension equal to $2\ell - 2$. Thus being a V-manifold, the space $N(\xi)$, if not smooth, is stratified in the usual way. We do not pursue this further.

3.7. Genus ≥ 2; $G = \mathrm{O}(3)$

The matrix $-\mathrm{Id} \in \mathrm{O}(3)$ is central and has order 2. Consequently the group $G = \mathrm{O}(3)$ may be written as a direct product $G = \mathrm{SO}(3) \times \mathbf{Z}/2$. Principal $\mathrm{O}(3)$-bundles ξ over Σ having a connected total space P are classified as follows: As in the previous Subsection, such a bundle ξ determines a unique homomorphism ϕ_ξ from π to $\mathbf{Z}/2$ which, since P is assumed connected, has to be *non-trivial*. Now, given a non-trivial homomorphism ϕ from π to $\mathbf{Z}/2$, there is always a principal bundle ξ so that $\phi = \phi_\xi$, and for fixed ϕ, the principal $\mathrm{O}(3)$-bundles having $\phi_\xi = \phi$ are classified by the cohomology group $\mathrm{H}^2(\Sigma, \pi_1(\mathrm{SO}(3)) \cong \mathbf{Z}/2$. Thus, given ϕ, there are two non-isomorphic principal $\mathrm{O}(3)$-bundles ξ having $\phi_\xi = \phi$. The direct product decomposition $\mathrm{O}(3) = \mathrm{SO}(3) \times \mathbf{Z}/2$ induces a decomposition of $\mathrm{Hom}(\pi, \mathrm{O}(3))$ into the direct product of $\mathrm{Hom}(\pi, \mathrm{SO}(3))$ and $\mathrm{Hom}(\pi, \mathbf{Z}/2)$. Hence the stratification of $N(\xi)$ may be obtained from the stratification of the corresponding moduli space for the corresponding $\mathrm{SO}(3)$-bundle described in (3.3) or (3.5) as appropriate, whatever principal $\mathrm{O}(3)$-bundle ξ. We leave the details to the reader.

References

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